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Construction of KP hierarchies in terms of finite number of fields and their abelianization

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The $2M$-boson representations of the KP hierarchy are constructed in terms of M mutually independent two-boson KP representations for arbitrary number M. Our construction establishes the multi-boson representations of the KP hierarchy as consistent Poisson reductions of the standard KP hierarchy within the R-matrix scheme. As a byproduct we obtain a complete description of any finitely-many-field formulation of the KP hierarchy in terms of Darboux coordinates with respect to the first Hamiltonian structure. This results in a series of representations of $\mathbf{W}_{1+\infty}$ algebra made out of an arbitrary even number of boson fields.

1. Introduction

It has been recognized in the last few years that the integrability structure appearing in the double scaling limit of the one-matrix model can be analyzed in terms of the KdV hierarchy augmented by the string equation [1]. This result created a lot of interest in various types of integrable hierarchies in connection with attempts to uncover similar pattern in the multi-matrix models. However, taking the continuum limit in the multi-matrix models encountered severe difficulties. An attempt to circumvent these problems was made in [2], where the matrix models were represented as discrete linear systems giving rise to lattice integrable hierarchies from which differential hierarchies were extracted without taking the continuum limit. In this approach the one-matrix model resulted in a differential hierarchy known as the two-boson KP hierarchy [3,4], which via the Dirac constraint mechanism reduces to a simple KdV hierarchy. In the case of multi-matrix models the same procedure [5] resulted in pseudo-differential operators, which formally generalized the Lax operator of the two-boson KP hierarchy. In view of the above development it is, therefore, natural to inquire about the precise status of these differential operators within the setting of KP hierarchy, especially, to prove the Hamiltonian nature of the corresponding flows.

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In [6] we addressed the question of linking and classifying the integrable systems falling into the general class of KP_l (with $l = 0, 1, 2$) hierarchies originating in the Adler–Kostant–Symes (AKS) construction [7]. As we show in this paper these considerations will prove to be essential for the aforementioned matrix model construction.

One of the features of our R-matrix coadjoint-orbit approach was that it singled out the two- and four-boson systems as two finite-dimensional [#1] field representations of KP hierarchy. As we point out in this paper, the four-boson system has a dual status: on the one hand – a finite-dimensional coadjoint orbit inside the $KP_{l=2}$ hierarchy, and on the other hand – a composite system consisting of two independent two-boson systems. This rises the question whether this picture could be extended, namely, whether two-boson systems could be used as building blocks of finitely-many-boson KP hierarchies fitting into the AKS formalism with Kirillov–Kostant R-Poisson bracket. These systems would then provide finitely-many-field representations of $\mathbf{W}_{1+\infty}$ algebras. We present in this paper an explicit construction of such systems consisting of an arbitrary finite even number of bosons. These systems are shown to be legitimate Poisson restrictions of the KP hierarchy. A crucial role in our approach is played by a recurrence relation connecting $2M$-boson and $2(M-1)$-boson KP systems. One can interpret our results as abelianization, meaning that two-boson KP hierarchies provide the Darboux coordinates for the many-boson representations of KP. Each $2M$-boson representation of KP within the first Hamiltonian structure is built-up out of M mutually commuting two-boson systems.

In section 2 we present the AKS formulation of three integrable systems of KP type and their equivalence via symplectic gauge transformations in a form, which yields the basis of our subsequent construction. More precisely, in the AKS setting there exist two consistent restrictions of the KP hierarchy in terms of two- and four-boson systems. Each of these two systems provides an example of a finitely-many-field representation of $\mathbf{W}_{1+\infty}$ algebra as it follows automatically by virtue of the symplectic character of the gauge transformations mapping these systems into the standard $KP_{l=0}$ hierarchy [6].

The main result of this paper is presented in section 3, where we prove that the new class of the multi-boson Lax operators constitutes a consistent Poisson reduction of the standard KP manifold with infinitely many fields. In particular, this results in a series of representations of $\mathbf{W}_{1+\infty}$ algebra made out of an arbitrary even number of boson fields. Also, the explicit abelianization formulas for the multi-boson Lax operators are written down. Our general construction is illustrated for the specific case of the six-boson KP hierarchy in section 4. We conclude by indicating in section 5 possible directions of future investigations.

2. Algebraic and geometric preliminaries

2.1. AKS approach to KP hierarchy. Symplectic gauge transformations

We recall first how the AKS [7] formalism associates three KP-type integrable systems labeled by the index $l = 0, 1, 2$ to three possible decompositions of the Lie algebra $\mathcal{G}$ of pseudo-differential operators on the circle into a linear sum of two subalgebras. Writing an arbitrary pseudo-differential operator $X \in \mathcal{G}$ as $X = \sum_{k \geqslant -\infty} D^k X_k(x)$ [#2] we can decompose $\mathcal{G}$ as $\mathcal{G} = \mathcal{G}^l_+ \oplus \mathcal{G}^l_-$ [8,3,6] with

$$\mathcal{G}^l_+ = \left\{ X_{\geqslant l} = \sum_{i=l}^{\infty} \mathrm{D}^i X_i(x) \right\}, \quad \mathcal{G}^l_- = \left\{ X_{<l} = \sum_{i=-l+1}^{\infty} \mathrm{D}^{-i} X_{-i}(x) \right\}, \tag{1}$$

for $l = 0, 1, 2$. The corresponding dual spaces with respect to the Adler bilinear pairing $\langle L \mid X \rangle = \mathrm{Tr}(LX) = \int \mathrm{d}x \, \mathrm{Res}(LX)$ are given by

#1 The terms "finite-dimensional" and "infinite-dimensional" refer to the number of functional (field) dimensions.
#2 Throughout the text D denotes the differential operator $\mathrm{D} = \partial/\partial x$, whereas derivative acting on a function will be denoted by $\partial_x f$.

$$\mathcal{G}_+^{l\,*} = \left\{ L_{<-l} = \sum_{i=l+1}^{\infty} u_{-i}(x)\mathrm{D}^{-i} \right\}, \quad \mathcal{G}_-^{l\,*} = \left\{ L_{\geqslant -l} = \sum_{i=-l}^{\infty} u_i(x)\mathrm{D}^{i} \right\}. \tag{2}$$

Note the opposite ordering of D's and coefficient functions in (1) and (2). Denoting the projections on the subalgebras in (1) by $\mathcal{P}_\pm^l$ we can define the R-matrix operator on $\mathcal{G}$ as $R_l \equiv \mathcal{P}_+^l - \mathcal{P}_-^l$. There exists a new Lie commutator on $\mathcal{G}$ associated to each R_l-matrix and defined by $[X,Y]_{R_l} \equiv \frac{1}{2}[R_l X, Y] + \frac{1}{2}[X, R_l Y] = [X_{\geqslant l}, Y_{\geqslant l}] - [X_{<l}, Y_{<l}]$. The Poisson structure on $\mathcal{G}^*$ follows now naturally by generalizing the Kirillov–Kostant formula to the R_l-commutator as follows:

$$\{F, H\}_{R_l}(L) = -\left\langle L \,\middle|\, [\nabla F(L), \nabla H(L)]_{R_l} \right\rangle ; \tag{3}$$

see [7,6] for details. The R_l-Poisson bracket (3) is the first Hamiltonian structure for the KP_l hierarchy.

Consider now Casimir functions on $\mathcal{G}^*$ defined as functions, which are invariant under coadjoint action of the corresponding Lie group G. The Casimir functions constitute a set of functions in involution on the Poisson manifold. A convenient choice of Casimirs is provided by $H_{n+1} = (1/(n+1))\,\mathrm{Tr}\,L^{n+1}$ for which $\nabla H_{n+1} = (L^n)_{\geqslant l}$. The Hamiltonian equations of motion on $(\mathcal{G}^*, \{\cdot,\cdot\}_{R_l})$ associated to these Casimir functions,

$$\frac{\partial L}{\partial t_n} = \{H_n, L\}_{R_l}, \tag{4}$$

take, according to (3), the form of Lax evolution equations on $\mathcal{G}^*$ for all three integrable KP_l systems:

$$\frac{\partial L}{\partial t_n} = \left[(L^n)_{\geqslant l}, L \right], \quad l = 0,1,2. \tag{5}$$

There is a way of relating Lax operators of different KP_l hierarchies by a map, which plays a role of gauge transformation. Consider first (5) with $l = 0$. It describes the standard KP flow equation with the Lax operator,

$$L \equiv \mathrm{D} + \sum_{i=1}^{\infty} u_i(x,t)\mathrm{D}^{-i}, \tag{6}$$

with the first Hamiltonian structure induced by the R_0-bracket (3) being $\{u_n(x), u_m(y)\}_{R_0} = \Omega^{(0)}_{n-1,m-1}(u(x)) \times \delta(x-y)$, where the Watanabe form on the right-hand side can be obtained from the general expression

$$\Omega^{(l)}_{nm}(u(x)) \equiv \sum_{k=0}^{n+l} (-1)^k \binom{n+l}{k} u_{n+m+l-k+1}(x)\mathrm{D}_x^k - \sum_{k=0}^{m+l} \binom{m+l}{k} \mathrm{D}_x^k u_{n+m+l-k+1}(x). \tag{7}$$

We call the integrable system characterized by $l = 1$ in (5) a $\mathrm{KP}_{l=1}$ hierarchy and associate to it a Lax operator as follows. Consider elements in $\mathcal{G}_-^{l=1\,*}$ (2) of the type $\widetilde{L}_1 = \mathrm{D} + u_0 + u_1\mathrm{D}^{-1}$, which preserve their form under R_1-coadjoint action, spanning therefore a R_1-orbit of finite field dimensions. A complete Lax operator is obtained by adding $\widetilde{L}_1$ to the general element L_- of $\mathcal{G}_+^{l=1\,*}$ (2):

$$L^{(l=1)} = \widetilde{L}_1 + L_- = \mathrm{D} + u_0 + u_1\mathrm{D}^{-1} + \sum_{i\geqslant 2} u_i\mathrm{D}^{-i}. \tag{8}$$

There is a map, resembling a gauge transformation, between the Lax operators L (6) and $L^{(l=1)}$ (8):

$$L \equiv G^{-1}L^{(l=1)}G = \mathrm{D} + \sum_{i=1}^{\infty} v_i\mathrm{D}^{-i}, \quad G \equiv \exp\left(-\int^x u_0\,\mathrm{d}x' \right). \tag{9}$$

Finally, we consider the $\mathrm{KP}_{l=2}$ hierarchy. Here elements of $\mathcal{G}_-^{l=2*}$ (2) of the form

$$\widetilde{L}_2 = u_{-1}\mathrm{D} + u_0 + u_1\mathrm{D}^{-1} + u_2\mathrm{D}^{-2} \tag{10}$$

span an invariant subspace under the coadjoint action induced by $R_{l=2}$-matrix. The complete Lax operator for $\mathrm{KP}_{l=2}$ is then given by $L^{(l=2)} = \widetilde{L}_2 + L_- = u_{-1}\mathrm{D} + u_0 + u_1\mathrm{D}^{-1} + u_2\mathrm{D}^{-2} + \sum_{i\geqslant 3} u_i\mathrm{D}^{-i}$ and transforms to the Lax operator of the $\mathrm{KP}_{l=1}$ hierarchy under the gauge transformation generated by the centerless Virasoro group. Explicitly we find [6]

$$\exp(\phi(x)\mathrm{D})L^{(l=2)}\exp(-\phi(x)\mathrm{D}) = \mathrm{D} + \tilde{u}_0 + \tilde{u}_1\mathrm{D}^{-1} + \tilde{u}_2\mathrm{D}^{-2} + \sum_{i\geqslant 3}\tilde{u}_i\mathrm{D}^{-i}, \tag{11}$$

where $\phi(x)$ is chosen in such a way that $u_{-1}\,(F_\phi(x)) = \partial_x F_\phi(x)$ with $F_\phi(x) = \exp(\phi(x)\partial_x)x$ representing a finite conformal transformation. Clearly, the Lax operator on the right-hand side of (11) belongs to $\mathrm{KP}_{l=1}$ hierarchy.

The main result of [6] was an explicit proof that the gauge transformations in (9) and (11) are symplectic maps, meaning that they map the R_l-Poisson bracket structure for KP_l to the $R_{l'}$-bracket structure for $\mathrm{KP}_{l'}$. This result established full gauge equivalence between all three integrable systems described by the KP_l hierarchies. We will illustrate this principle for the finite-dimensional cases of two- and four-boson systems associated with thr $\widetilde{L}_1$ and $\widetilde{L}_2$ (10) operators.

3. Two- and four-boson representations of KP and $\mathbf{W}_{1+\infty}$ and their relation

The starting point is the two-boson Lax operator $\widetilde{L}_1 = \mathrm{D} + b + a\mathrm{D}^{-1}$ in the $\mathrm{KP}_{l=1}$ hierarchy (notations for Lax coefficients are changed for later convenience). The corresponding R_1-bracket reads $\{a(x), b(y)\}_{R_1} = -\partial_x\delta(x-y)$ and zero otherwise.

Under the gauge transformation

$$L_1 \equiv \exp\left(\int b\right)\widetilde{L}_1\exp\left(-\int b\right) = \mathrm{D} + a\,(\mathrm{D}-b)^{-1} = \mathrm{D} + \sum_{n=0}^{\infty}(-1)^n aP_n(-b)\mathrm{D}^{-1-n}, \tag{12}$$

$\widetilde{L}_1$ transforms to the constrained $\mathrm{KP}_{l=0}$ Lax operator L_1 with coefficients $u_{n+1} = (-1)^n aP_n(-b)$ given in terms of the Faá di Bruno polynomials $P_n(b) \equiv (\partial + b)^n \cdot 1$. As a consequence of the symplectic character of the gauge transformation in (12) we find therefore

$$\Big\{(-1)^n aP_n(-b)\,,\,(-1)^m aP_m(-b)\Big\}_{R_0} = \Omega^{(0)}_{nm}\Big(u_{n+1} = (-1)^n aP_n(-b)\Big)\,\delta(x-y) \tag{13}$$

and hence the two-boson system realizes the $\mathbf{W}_{1+\infty}$ algebra.

Similar remarks apply to the restricted KP system of four bosons described by the Lax operator $\widetilde{L}_2$ (10) of the $\mathrm{KP}_{l=2}$ hierarchy. The relevant gauge transformation (11) acts now as follows ($F'_\phi \equiv \partial_x F_\phi$):

$$\begin{aligned}\exp(\phi(x)\mathrm{D})\widetilde{L}_2\exp(-\phi(x)\mathrm{D}) &= \mathrm{D} + u_0\,(F_\phi(x)) + u_1\,(F_\phi(x))\,\mathrm{D}^{-1}F'_\phi(x) + u_2\,(F_\phi(x))\,\mathrm{D}^{-1}F'_\phi(x)\mathrm{D}^{-1}F'_\phi(x)\\ &= \exp(-\ln F'_\phi(x))\left(\mathrm{D} + B_2 + A_2\mathrm{D}^{-1} + A_1(\mathrm{D}-\overline{B}_1)^{-1}\mathrm{D}^{-1}\right)\exp(\ln F'_\phi(x)).\end{aligned} \tag{14}$$

Therefore, it connects $\widetilde{L}_2$ via additional Abelian gauge transformation to the $\mathrm{KP}_{l=1}$ Lax operator

$$\widehat{L}_2 = \mathrm{D} + B_2 + A_2\mathrm{D}^{-1} + A_1(\mathrm{D}-\overline{B}_1)^{-1}\mathrm{D}^{-1}, \tag{15}$$

whose coefficient fields

$$A_1 = u_2\left(F_\phi(x)\right)\left(F_\phi'(x)\right)^2, \quad \overline{B}_1 = -\partial_x \ln F_\phi'(x),$$
$$A_2 = u_1\left(F_\phi(x)\right)F_\phi'(x), \quad B_2 = u_0\left(F_\phi(x)\right) - \partial_x \ln F_\phi'(x), \tag{16}$$

satisfy according to [6] the following algebra:

$$\{A_2(x), B_2(y)\} = -\partial_x\delta(x-y), \tag{17}$$
$$\left\{A_1(x), \overline{B}_1(y)\right\} = -\left(\partial_x + \overline{B}_1(x)\right)\partial_x\delta(x-y), \tag{18}$$
$$\{A_1(x), A_1(y),\} = -2A_1(x)\partial_x\delta(x-y) - (\partial_x A_1)\delta(x-y), \tag{19}$$

as follows from the original R_2-Poisson brackets (3) for u_{-1}, u_0, u_1, u_2 within the $\mathrm{KP}_{l=2}$ hierarchy.

In order to end up with the Lax operator in $\mathrm{KP}_{l=0}$ we then apply to (15) a gauge transformation generated by $-\int B_2$ with the result

$$L_2 = \exp\left(\int B_2\right)\widehat{L}_2\exp\left(-\int B_2\right) = \mathrm{D} + A_2(\mathrm{D}-B_2)^{-1} + A_1(\mathrm{D}-B_1)^{-1}(\mathrm{D}-B_2)^{-1}$$
$$= \mathrm{D} + \sum_{n=0}^{\infty}(-1)^n A_2 P_n(-B_2)\mathrm{D}^{-1-n} + \sum_{n=0}^{\infty} A_1 P_n^{(2)}(B_2,B_1)\mathrm{D}^{-2-n} \equiv \mathrm{D} + \sum_{k=1}^{\infty} U_k[A_{1,2},B_{1,2}]\mathrm{D}^{-k}, \tag{20}$$

where $B_1 \equiv \overline{B}_1 + B_2$ and where we have introduced the double Faá di Bruno polynomials $P_n^{(2)}(B_2,B_1) = \sum_{l,k\geqslant 0}^{l+k=n}(-\partial + B_1)^l(-\partial + B_2)^k \cdot 1$ (cf. eqs. (44)–(51) in ref. [6]). On the basis of a theorem [6] about the symplectic character of both types of gauge transformations used in (14) and (20) we know that coefficient fields of L_2 from (20) satisfy the Poisson algebra

$$\left\{U_n[A_{1,2},B_{1,2}](x), U_m[A_{1,2},B_{1,2}](y)\right\}_{R_0} = \Omega^{(0)}_{n-1,m-1}\left(U_k[A_{1,2},B_{1,2}]\right)\delta(x-y), \tag{21}$$

whenever $A_{1,2}$, $B_{1,2}$ satisfy (17)–(19) and, therefore, the four-boson system forms a representation of $\mathbf{W}_{1+\infty}$ algebra.

As already observed in [6] the four-boson Poisson algebra (17)–(19) decomposes into a direct sum of Heisenberg algebra generated by the two-boson system (A_2, B_2) and separates the algebra of coupled spin-2 and spin-1 fields $(A_1, \overline{B}_1)$. It is well-known (see for instance [9]) that there exists in the KP setting a generalized Miura transformation, which maps elements of the Heisenberg algebra to the higher spin algebras. In the case of $(A_1, \overline{B}_1)$ fields and their algebra the generalized Miura transformation takes the following form:

$$A_1 = (\partial + b_1)a_1, \quad \overline{B}_1 = b_1, \tag{22}$$

in terms of the two-boson system (a_1, b_1) satisfying the Heisenberg algebra $\{a_1(x), b_1(y)\} = -\partial_x\delta(x-y)$. Summarizing we can say that the four-boson KP system given by (20) can be abelianized in terms of two mutually Poisson-commuting two-boson systems (a_1, b_1) and (a_2, b_2) entering into the generalized Miura transformation,

$$A_2 = a_2, \quad A_1 = (\partial + b_1)a_1, \quad B_2 = b_2, \quad B_1 = b_1 + b_2, \tag{23}$$
$$\{a_i(x), b_j(y)\} = -\delta_{ij}\partial_x\delta(x-y), \quad i,j = 1,2, \tag{24}$$

which reproduces the algebra (17)–(19). The application of the generalized Miura transformation (23) can be visualized as a recurrence relation connecting the two-boson $L_1 \equiv \mathrm{D} + a_1(\mathrm{D} - b_1)^{-1}$ and four-boson L_2 (20) Lax operators. Using (23), we find by simple calculation

$$L_2 = \exp\left(\int b_2\right)\left[b_2 + (a_2 - a_1)\mathrm{D}^{-1} + \mathrm{D}L_1\mathrm{D}^{-1}\right]\exp\left(-\int b_2\right). \tag{25}$$

This recurrence relation connects two- and four-boson Lax operators by an Abelian gauge transformation and the dressing operation $\mathrm{D}L\mathrm{D}^{-1}$. This will remain a general feature in section 3 when we address the problem of building multi-boson KP hierarchies out of any number of independent two-boson systems. We will use there this relation to study the Poisson bracket algebra of the composite systems.

4. Poisson reduction

The four-boson representation of the KP hierarchy admits an alternative description in terms of Poisson reduction on the phase space of general Lax operators (6). It is precisely the Poisson reduction scheme which provides the proper basis for our generalization of the previous construction of two-boson and four-boson KP representations to representations of KP in terms of an arbitrary finite number of boson field pairs.

First, let us recall some general notions [10]. Let $(\mathcal{M}, P)$ be a smooth Poisson [#3] manifold with Poisson structure $P: T^*(\mathcal{M}) \longrightarrow T(\mathcal{M})$. In local coordinates $\{x^i\}_{i=1}^{\dim \mathcal{M}}$ on $\mathcal{M}$ the Poisson bracket of arbitrary smooth functions, defined by the Poisson structure P, is given as

$$\{f, g\}_P = \langle P\nabla f \mid \nabla g\rangle = \omega^{ij}(x)\,\frac{\partial f}{\partial x^i}\,\frac{\partial g}{\partial x^j}, \tag{26}$$

where the angle brackets denote pairing between $T^*(\mathcal{M})$ and $T(\mathcal{M})$.

Let S be a smooth submanifold of $\mathcal{M}$ with local coordinates $\{\sigma^\alpha\}_{\alpha=1}^{\dim S}$ and embedding $\mu: S \longrightarrow \mathcal{M}$. Now, a Poisson structure $P': T^*(S) \longrightarrow T(S)$ on $S \subset \mathcal{M}$ is called *Poisson reduction* of P if for arbitrary functions on $\mathcal{M}$ the following property is satisfied:

$$\mu^*\left(\{f, g\}_P\right) = \{\mu^* f, \mu^* g\}_{P'}. \tag{27}$$

In other words, restriction of the Poisson brackets with respect to P of arbitrary functions on $\mathcal{M}$ to the submanifold S is equivalent to computing the Poisson brackets with respect to P' of the restrictions on S of these same functions [#4].

In local coordinates eq. (27) can be written as (recall $\mu^* f(\sigma) = f\left(x(\sigma)\right)$):

$$\omega^{ij}\left(x(\sigma)\right) = \widehat{\omega}^{\alpha\beta}(\sigma)\,\frac{\partial x^i}{\partial \sigma^\alpha}\,\frac{\partial x^j}{\partial \sigma^\beta}\quad\left(= \left\{x^i(\sigma), x^j(\sigma)\right\}_{P'}\right), \tag{28}$$

where $\widehat{\omega}^{\alpha\beta}(\sigma)$ is the Poisson tensor of P'.

Comparing (28) with (21) and identifying $x^i \sim L$ from (6), $\sigma^\alpha \sim (A_{1,2}, B_{1,2})$ and $x^i(\sigma) \sim L_2$ from (20), it is readily seen that four-boson representation of KP (20), (21) is indeed a genuine Poisson reduction

[#3] $\mathcal{M}$ needs not to be symplectic manifold, i.e., the Poisson structure P might be degenerate (the Poisson tensor $\omega^{ij}(x)$ being non-invertible).

[#4] Let us stress that, in the case when $(\mathcal{M}, P)$ is symplectic, the *Poisson* reduction P' of the Poisson structure P is in general different from the *Dirac* reduction thereof. The associated Dirac brackets are of the form $\{\mu^* f, \mu^* g\}_{\mathrm{DB}} = \mu^*(\{f, g\}_P - \{f, \Psi_A\}_P (C^{-1})^{AB}\{\Psi_B, g\}_P)$, where $S = \{\Psi_A = 0\}$ is defined through the set of Dirac second class constraints Ψ_A and $C^{AB} = \{\Psi_A, \Psi_B\}_P$.

of the original Kirillov–Kostant R_0-Poisson structure P (3) on the infinite-dimensional Lax manifold $\mathcal{M} = \{L = \mathrm{D} + \sum_{k=1}^{\infty} u_k(x)\mathrm{D}^{-k}\}$ to the Poisson structure P' (17)–(19) on the finite-dimensional manifold $S = \{L_2\,(A_{1,2}, B_{1,2})\}$ (20). Obviously, a similar remark applies as well to the two-boson KP system.

5. KP and $\mathbf{W}_{1+\infty}$ in terms of $2M$ fields for arbitrary M

Let us consider the sequence of pseudo-differential operators obtained from the natural generalization of the recursive relation eq. (25) for arbitrary $M = 2, 3, ...,$

$$L_M \equiv L_M(a,b) \equiv L_M\,(a_1, b_1; ...; a_M, b_M)\,,$$

$$L_M = \exp\left(\int b_M\right)\left[b_M + (a_M - a_{M-1})\mathrm{D}^{-1} + \mathrm{D}L_{M-1}\mathrm{D}^{-1}\right]\exp\left(-\int b_M\right), \tag{29}$$

where L_1 and L_2 are the two- and four-boson KP operators, respectively, (eqs. (12) and (20), or (25)), and the boson fields $(a_r, b_r)_{r=1}^{M}$ span the Heisenberg Poisson bracket algebra:

$$\{a_r(x), b_s(y)\}_{P'} = -\delta_{rs}\partial_x\delta(x-y)\,. \tag{30}$$

Using the identities $\exp(\int b_M)\mathrm{D}^{\pm 1}\exp(-\int b_M) = (\mathrm{D} - b_M)^{\pm 1}$, eq. (29) can be rewritten in an equivalent form, similar to expression (20) for L_2, which is valid for any M:

$$L_M = \mathrm{D} + \sum_{l=1}^{M} A_l^{(M)}\left(\mathrm{D} - B_l^{(M)}\right)^{-1}\left(\mathrm{D} - B_{l+1}^{(M)}\right)^{-1}\cdots\left(\mathrm{D} - B_M^{(M)}\right)^{-1}, \tag{31}$$

where the coefficient fields satisfy the simple recursion relations

$$A_M^{(M)} = a_M\,,\quad B_M^{(M)} = b_M\,,\quad B_l^{(M)} = b_M + B_l^{(M-1)}\quad (l = 1,2,...,M-1)\,,$$

$$A_1^{(M)} = \left(\partial + B_1^{(M-1)}\right)A_1^{(M-1)},\quad A_l^{(M)} = A_{l-1}^{(M-1)} + \left(\partial + B_l^{(M-1)}\right)A_l^{(M-1)}\quad (l = 2,...,M-1)\,. \tag{32}$$

These recursion relations can be easily solved in terms of the free fields a_r, b_r from (30) to yield

$$B_l^{(M)} = \sum_{s=l}^{M} b_s\,,\quad A_M^{(M)} = a_M\,, \tag{33}$$

$$A_{M-r}^{(M)} = \sum_{n_r=r}^{M-1}\cdots\sum_{n_2=2}^{n_3-1}\sum_{n_1=1}^{n_2-1}(\partial + b_{n_r} + ... + b_{n_r-r+1})\cdots\left(\partial + b_{n_2} + b_{n_2-1}\right)(\partial + b_{n_1})\,a_{n_1}\,. \tag{34}$$

The coefficients of the pseudo-differential operator (29) (or (31)) have the following explicit expressions:

$$L_M = \mathrm{D} + \sum_{k=1}^{\infty} U_k[(a,b)](x)\mathrm{D}^{-k}\,, \tag{35}$$

$$U_k[(a,b)](x) = a_M P_{k-1}^{(1)}(b_M) + \sum_{r=1}^{\min(M-1,k-1)} A_{M-r}^{(M)} P_{k-1-r}^{(r+1)}\left(b_M, b_M + b_{M-1}, ..., \sum_{l=M-r}^{M} b_l\right), \tag{36}$$

where $A^{(M)}_{M-r}$ are the same as in (34), and $P^{(N)}_n$ denote the (multiple) Faá di Bruno polynomials

$$P^{(N)}_n(B_N, B_{N-1}, ..., B_1) = \sum_{m_1+...+m_N=n} (-\partial + B_1)^{m_1} \cdots (-\partial + B_N)^{m_N} \cdot 1 . \tag{37}$$

The main result of this section is contained in the following

Theorem. The $2M$-field Lax operators (29) (or (31)) are consistent Poisson reductions of the general KP Lax operator (6) for any $M = 1, 2, 3, \dots$.

In other words, we shall prove that the Heisenberg Poisson bracket algebra P' (30) for $(a_r, b_r)^M_{r=1}$ *implies* the following Poisson brackets for L_M from (31) or (29) (recall eqs. (27) and (28)):

$$\left\{\left\langle L_M \mid X\right\rangle , \left\langle L_M \mid Y\right\rangle\right\}_{P'} = -\left\langle L_M \mid [X, Y]\right\rangle , \tag{38}$$

where X, Y are arbitrary fixed elements of the algebra of pseudo-differential operators and $\langle\cdot|\cdot\rangle$ indicates the Adler bilinear pairing.

Remark. Let us particularly stress that the truncation of the form of the general Lax operator (6) within the original KP Poisson brackets (3), leading to (38), may *not* be necessarily consistent. Namely, it does *not* automatically guarantee the closure of the infinite number of Poisson brackets for the *infinite* number of Lax coefficient fields $U_k[(a,b)](x)$ (36) as functionals of the *finite* number of independent fields (a_r, b_r) with respect to their fundamental Poisson brackets (30).

Thus, the present proof that eq. (38) is a consistent Poisson reduction (cf. subsection 2.3) provides the principle ingredient in the construction of integrable Hamiltonian systems which are representations of KP hierarchies in terms of finite number of fields [#5].

The proof of (38) proceeds by induction in M. It has already been established for $M = 1$ [3,4] and $M = 2$ [6]. Now, let us assume that (38) is valid for L_{M-1} and rewrite $\left\langle L_M \mid X\right\rangle$ in the form

$$\left\langle L_M \mid X\right\rangle = \int \mathrm{d}x\ (a_M - a_{M-1})(x) X_{(0)}(b_M)(x) + \left\langle L_{M-1} \mid \left(\mathrm{D}^{-1} X(b_M)\mathrm{D}\right)_+\right\rangle , \tag{39}$$

$$X(b_M) \equiv \exp\left(-\int b_M\right) X \exp\left(\int b_M\right), \tag{40}$$

where the subscripts (+) and (0) denote purely differential and zero order part of the corresponding (pseudo-) differential symbol. Substituting (39) into the left-hand side of (38) we get

$$\begin{aligned}
&\left\{\left\langle L_M \mid X\right\rangle , \left\langle L_M \mid Y\right\rangle\right\}_{P'} = \int\int \mathrm{d}x\,\mathrm{d}y\ \left\{X_{(0)}(b_M)(x)\,(a_M - a_{M-1})(x) , Y_{(0)}(b_M)(y)\,(a_M - a_{M-1})(y)\right\}_{P'} \\
&+ \left\{\left\langle L_{M-1} \mid \left(\mathrm{D}^{-1} X(b_M)\mathrm{D}\right)_+\right\rangle , \left\langle L_{M-1} \mid \left(\mathrm{D}^{-1} Y(b_M)\mathrm{D}\right)_+\right\rangle\right\}_{P'} \\
&+ \int \mathrm{d}y \Big(X_{(0)}(b_M)(y) \Big[\left\langle L_{M-1} \mid \left(\mathrm{D}^{-1} \{a_M(y), Y(b_M)\}_{R'} \mathrm{D}\right)_+\right\rangle \\
&- \left\langle \{a_{M-1}(y), L_{M-1}\}_{P'} \mid \left(\mathrm{D}^{-1} Y(b_M)\mathrm{D}\right)_+\right\rangle\Big] - \left(X(b_M) \longleftrightarrow Y(b_M)\right)\Big).
\end{aligned} \tag{41}$$

[#5] Flow equations with Lax operators of the form (31) recently appeared in the study of multi-matrix models [5]. Our theorem proves that these flows are Hamiltonian ones.

Using (30) and (40) one can easily check the identities

$$\{a_M(y), X(b_M)\}_{P'} = -[\delta(y-x), X(b_M)], \tag{42}$$

$$\{a_{M-1}(y), L_{M-1}\}_{P'} = \left[\delta(y-x), (L_{M-1})_-\right], \tag{43}$$

where the right-hand side of (42) and (43) indicate (pseudo-) differential operator commutators with respect to x, and the subscript $(-)$ denotes purely pseudo-differential part.

Using the induction hypothesis for the second term in the right-hand side of (41) and substituting (42) and (43) into (41) we obtain

$$\begin{aligned}
\left\{\langle L_M \mid X\rangle, \langle L_M \mid Y\rangle\right\}_{P'} &= -\int dx\,(a_M - a_{M-1})(x)\,[X(b_M), Y(b_M)]_{(0)}(x) \\
&- \Big\langle L_{M-1}\Big| \left[\left(D^{-1}X(b_M)D\right)_+, \left(D^{-1}Y(b_M)D\right)_+\right] \\
&+ \left(D^{-1}\left(\left[X_{(0)}(b_M), Y(b_M)\right] + \left[X(b_M), Y_{(0)}(b_M)\right]\right)D\right)_+ \\
&- \left[X_{(0)}(b_M), \left(D^{-1}Y(b_M)D\right)_+\right] - \left[\left(D^{-1}X(b_M)D\right)_+, Y_{(0)}(b_M)\right]\Big\rangle \\
&= -\int dx\,(a_M - a_{M-1})(x)\,[X(b_M), Y(b_M)]_{(0)}(x) - \left\langle L_{M-1} \mid \left[D^{-1}X(b_M)D, D^{-1}Y(b_M)D\right]_+\right\rangle \\
&= -\langle L_M \mid [X, Y]\rangle,
\end{aligned} \tag{44}$$

where in the last equality once again representation (39) was used. This completes the proof of our main statement about the consistency of the KP Poisson reduction (38).

Let us point out the following important observation. Eqs. (33) and (34) are nothing but *abelianization* of the $2M$-field KP hierarchy given by (31). Namely, all coefficients $U_k[(a,b)](x)$ of the $2M$-field KP Lax operator (35) are explicitly expressed (see eq. (36)) in terms of M pairs of free fields $(a_r, b_r)_{r=1}^M$ satisfying the Heisenberg Poisson bracket algebra (30). From general Hamiltonian point of view $(a_r, b_r)_{r=1}^M$ can be viewed as Darboux canonical coordinates on the phase space of the $2M$-field KP system.

Furthermore, eq. (38) provides us with explicit (Poisson bracket) realization of $\mathbf{W}_{1+\infty}$ algebra in terms of $2M$ bosons for any $M = 1, 2, 3, \ldots$. Indeed, according to (38) the coefficient fields (35) satisfy, as functionals of $(a_r, b_r)_{r=1}^M$, the $\mathbf{W}_{1+\infty}$ Poisson bracket algebra with respect to P' (30):

$$\left\{U_k[(a,b)](x), U_l[(a,b)](y)\right\}_{P'} = \Omega^{(0)}_{k-1,l-1}\left(U[(a,b)]\right)\delta(x-y), \tag{45}$$

where $\Omega^{(0)}_{kl}$ is given in (7).

6. Example: six-boson KP hierarchy

In this section we shall specialize the general formulae of the previous section to represent the KP hierarchy in term of six-boson fields. The Lax operator (31) for $M = 3$ is

$$L_{(3)} = D + A_3(D-B_3)^{-1} + A_2(D-B_2)^{-1}(D-B_3)^{-1} + A_1(D-B_1)^{-1}(D-B_2)^{-1}(D-B_3)^{-1}. \tag{46}$$

It is abelianized by the substitutions (cf. (33), (34))

$$B_1 = b_1 + b_2 + b_3\,, \quad B_2 = b_2 + b_3\,, \quad B_3 = b_3\,, \tag{47}$$

$$A_1 = (\partial + b_1 + b_2)(\partial + b_1)a_1\,, \quad A_2 = (\partial + b_1)a_1 + (\partial + b_2)a_2\,, \quad A_3 = a_3\,. \tag{48}$$

The fields $(A_r, B_r)_{r=1}^3$ satisfy the Poisson bracket algebra (below we use notations $\overline{B}_{1,2} = B_{1,2} - B_3$):

$$\{A_3(x), B_3(x)\} = -\partial_x \delta(x-y)\,, \tag{49}$$

$$\{A_2(x), A_2(y)\} = -2A_2(x)\partial_x\delta(x-y) - (\partial_x A_2)\,\delta(x-y)\,, \tag{50}$$

$$\{A_2(x), A_3(y)\} = -3A_3(x)\partial_x\delta(x-y) - 2\,(\partial_x A_3)\,\delta(x-y)\,, \tag{51}$$

$$\left\{A_2(x), \overline{B}_2(y)\right\} = -\left(\partial_x + \overline{B}_2(x)\right)\partial_x\delta(x-y)\,, \tag{52}$$

$$\left\{A_2(x), \overline{B}_1(y)\right\} = -\left(2\partial_x + \overline{B}_1(x)\right)\partial_x\delta(x-y)\,, \tag{53}$$

$$\left\{A_1(x), \overline{B}_1(y)\right\} = -\left(\partial_x + \overline{B}_1\right)\left(\partial_x + \left(\overline{B}_1 - \overline{B}_2\right)\right)\partial_x\delta(x-y)\,, \tag{54}$$

$$\begin{aligned}\{A_1(x), A_1(y)\} &= A_1(x)\left(\partial_x - \overline{B}_1\right)^2\delta(x-y) - \left(\partial_x + \overline{B}_1\right)^2 A_1\,\delta(x-y) \\ &\quad + \left(\partial_x\left(A_1\overline{B}_2\right)\right)\delta(x-y) + 2A_1\overline{B}_2\,\partial_x\delta(x-y)\,.\end{aligned} \tag{55}$$

Finally, the $\mathbf{W}_{1+\infty}$ fields in the six-boson realization $(A_r, B_r)_{r=1}^3$ or, equivalently in terms of the Darboux fields $(a_r, b_r)_{r=1}^3$, read

$$U_1 = A_3 = a_3 \quad \text{(spin 1)}\,, \tag{56}$$

$$U_2 = A_3B_3 + A_2 = a_3b_3 + (\partial + b_1)a_1 + (\partial + b_2)a_2 \quad \text{(spin 2)}\,, \tag{57}$$

$$\begin{aligned}U_s &= A_3P^{(1)}_{s-1}(B_3) + A_2P^{(2)}_{s-2}(B_3, B_2) + A_1P^{(3)}_{s-3}(B_3, B_2, B_1) \\ &= a_3P^{(1)}_{s-1}(b_3) + \left[(\partial + b_1)a_1 + (\partial + b_2)a_2\right]P^{(2)}_{s-2}(b_3, b_3 + b_2) \\ &\quad + \left[(\partial + b_2 + b_1)(\partial + b_1)a_1\right]P^{(3)}_{s-3}(b_3, b_3 + b_2, b_3 + b_2 + b_1) \quad (s \geqslant 3)\,,\end{aligned} \tag{58}$$

where again we used the multiple Faá di Bruno polynomials (37).

7. Outlook and discussion

The strategy of this paper was to start with the two-boson KP system and work out the higher-boson representations by a recurrence procedure. In this way we achieved a description of multi-boson KP hierarchies through abelianization of the first Poisson structure. The simplicity of the final result rises hopes for future applications of our method. Let us briefly indicate possible directions of future investigations. The corner stone of our construction, the two-boson KP hierarchy, has recently been a subject of a quantization attempt [11] promoting the classical relation (in fact, gauge equivalence) with the non-linear Schrödinger (NLS) hierarchy to the quantum case. It seems natural to expect that one can extend this quantization procedure to four-, six-, etc. boson systems adding successively quantum NLS hierarchies according to our recurrence relation.

Another question, naturally arising from our analysis, is whether the result we obtained could be used to gain a new understanding of the lattice hierarchies connected with the multi-matrix models. In parallel to our observation in this paper one could expect a convenient redefinition of fields in the lattice hierarchies similar to the abelianized representation of the pseudo-differential multi-boson KP Lax operators.

We point out that the abelianization construction (Darboux coordinates) works so far only in the context of the first KP Hamiltonian structure. In view of the existence of a compatible second bracket structure in the unconstrained KP hierarchy, it will be interesting to study how the latter is affected by the Poisson reduction and what possible form the abelianization will take in this framework.

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